

**PICARD NUMBERS IN A FAMILY
OF HYPERKÄHLER MANIFOLDS
- A SUPPLEMENT TO THE ARTICLE OF R. BORCHERDS,
L. KATZARKOV, T. PANTEV, N. I. SHEPHERD-BARRON**

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Dedicated to Professor Tetsuji Shioda on the occasion of his sixtieth birthday

ABSTRACT. We remark the density of the jumping loci of the Picard number of a hyperkähler manifold under small one-dimensional deformation and provide some applications for the Mordell-Weil groups of Jacobian K3 surfaces.

§0. INTRODUCTION

Although the Néron-Severi groups and the Mordell-Weil groups are described by means of cohomology groups of abelian sheaves, the sheaves are neither coherent nor topological and not much are known about their behaviours under deformation.

In their article “Families of K3 surfaces”, R. Borchers, L. Katzarkov, T. Pantev and N. I. Shepherd-Barron have found the following phenomenon by studying the behaviour of the Picard numbers of K3 surfaces under *global* deformation:

Theorem ([BKPS]). *Any smooth complete family of minimal Kähler surfaces of Kodaira dimension 0 and constant Picard number is isotrivial.* \square

Their proof is a global argument based on the ampleness of the zero locus of an automorphic form on a relevant moduli space of K3 surfaces and therefore requires the completeness of the base space in essence.

The aim of this note is to remark that their Theorem can be immediately generalized to a *local one-dimensional* family of hyperkähler manifolds if one adopts a different approach (Main Theorem below together with Section 1). The idea is extremally simple: In stead of studying behaviours of the Picard groups in family directly, we consider deformation of the perpendicular part, i.e. periods. The validity of this reduction is ensured by the Lefschetz $(1, 1)$ -Theorem and the local Torelli Theorem for hyperkähler manifolds. We also provide some applications for the Mordell-Weil groups of Jacobian K3 surfaces (Section 2).

A hyperkähler manifold is by the definition a simply connected, compact Kähler manifold F with $H^{2,0}(F) = \mathbb{C}\omega_F$, where ω_F is everywhere non-degenerate. According to the Bogomolov decomposition Theorem [Be], these are one of the building

blocks of manifolds with trivial first Chern class. In this terminology, a K3 surface is nothing but a hyperkähler manifold of dimension 2. Due to the fundamental work by Bogomolov [Bo] and Beauville [Be], the following results hold for a hyperkähler manifold F of any dimension:

- (1) The Kuranishi space of F is smooth and universal;
- (2) There exists a primitive integral non-degenerate symmetric bilinear form $(*, *)$ on $H^2(F, \mathbb{Z})$ which induces on $H^2(F, \mathbb{C}) = H^{1,1}(F) \oplus \mathbb{C}\omega_F \oplus \mathbb{C}\overline{\omega}_F$ the Hodge structure of weight two and is of index $(3, B_2(F) - 3)$;
- (3) The local Torelli Theorem holds for the period map given by the Hodge structure on $H^2(F, \mathbb{Z})$ defined in (2).

Besides original articles, we also refer the readers to [Hu1, Section 1] as an excellent survey about these basics.

In this note, we consider a smooth family of hyperkähler manifolds $f : \mathcal{X} \rightarrow \Delta$ over a disk Δ . In this setting, the following two statements are equivalent:

- (1) f is trivial as a family, i.e. isomorphic to the product $F \times \Delta$ over Δ ;
- (2) all the fibers of f are isomorphic.

This equivalence is a direct consequence of the local Torelli Theorem and the universality of the Kuranishi space together with the fact that $\pi_1(\Delta) = \{1\}$.

We denote by $\rho(F)$ the Picard number of F , i.e. the rank of the Néron-Severi group $NS(F) := \text{Im}(c_1 : H^1(F, \mathcal{O}_F^\times) \rightarrow H^2(F, \mathbb{Z})) = (\mathbb{C}\omega_F)^\perp \cap H^2(F, \mathbb{Z})$. Here the last equality is due to the Lefschetz (1, 1)-Theorem. Note also that $0 \leq \rho(F) \leq N := B_2(F) - 2$.

Our main Theorem is as follows:

Main Theorem. *Let $f : \mathcal{X} \rightarrow \Delta$ be a non-trivial family of hyperkähler manifolds. Set $M := \min \{\rho(\mathcal{X}_t) | t \in \Delta\}$ and $\mathcal{S} := \{t \in \Delta | \rho(\mathcal{X}_t) > M\}$. Then, \mathcal{S} is a dense countable subset of Δ in the classical topology.*

The next simple example will illustrate the phenomenon described in the main Theorem fairly well:

Example. Let us denote by E_t the elliptic curve of period t . Let Δ be a small disk in the upper half plane \mathbb{H} . Then, one has a family of elliptic curves $h : \mathcal{E} \rightarrow \Delta$ with the level two structure such that $\mathcal{E}_t = E_t$. Taking a crepant resolution of the quotient of the product $g : \mathcal{E} \times E_{\sqrt{-1}} \rightarrow \Delta$ by the inversion, one obtains a family of K3 surfaces $f : \mathcal{X} \rightarrow \Delta$ such that $\mathcal{X}_t = \text{Km}(E_t \times E_{\sqrt{-1}})$. This family f satisfies $\rho(\mathcal{X}_t) = 20$ for $t \in \mathbb{Q}(\sqrt{-1})$ and $\rho(\mathcal{X}_t) = 18$ for $t \notin \mathbb{Q}(\sqrt{-1})$. In this example, we have $\mathcal{S} = \Delta \cap \mathbb{Q}(\sqrt{-1})$. \square

It is an easy fact that \mathcal{S} is at most countable and that the locus of the constant Picard number $\Delta - \mathcal{S}$ is dense and uncountable (and therefore “much bigger” than \mathcal{S}). The essential part of the main Theorem is in the converse: *existence of enough jumping points*.

As one of applications, we shall solve the following filling up problem of possible Picard numbers:

Corollary. *Let F be a hyperkähler manifold with $B_2(F) = N + 2$. Then, for each $0 \leq j \leq N$, there exists a hyperkähler manifold F_j such that F and F_j are deformation equivalent and that $\rho(F_j) = j$.*

Although some form of this kind of density results should be known for some experts at least in the case of K3 surfaces (See Acknowledgement of [BKPS]), as far as the author knows, there are no literatures in which an explicit statement and proof are given. We should remark that there are several possible forms of density results and that the density of the jumping points in a disk implies the absence of positive dimensional analytic subsets in the Kuranishi space over which the fibers are of constant Picard number. Therefore, our statement is stronger than the density of the jumping points both in the Kuranishi space and in positive dimensional global base spaces. For this remark, it might be worth noticing the following:

Fact. Given a family of K3 surfaces $f : \mathcal{X} \rightarrow \mathcal{B}$, we call a point $b \in \mathcal{B}$ a Kummer point if the fiber \mathcal{X}_b is isomorphic to a Kummer surface. Then:

- (1) As well-known, the set of Kummer points is dense in the base if the family is the Kuranishi family of a K3 surface. (See for instance [BPV].)
- (2) However, the one dimensional smooth non-trivial family of elliptic K3 surfaces $f : \mathcal{X} \rightarrow \Delta_t$ ($|t| \ll 1$) defined by the Weierstrass equation $y^2 = x^3 + x + (u^{11} - t)$ (over Δ_t) has no Kummer points. Indeed, since each fiber \mathcal{X}_t has non-symplectic automorphism of order 11 (given by $(x, y, u) \mapsto (x, y, \zeta_{11}u)$), the rank of the transcendental lattice must be divisible by $\varphi(11) = 10$ and one has then $\rho(\mathcal{X}_t) < 16$. (See [OZ, Example 2].) \square

This stronger version is also needed in the proof of the Corollary as well as in our application for the Mordell-Weil groups of Jacobian K3 surfaces.

It would be interesting to ask a similar question for complex tori of dimension ≥ 3 . We should also notice that a similar question for Calabi-Yau manifolds does not make much sense, because $\text{Pic}(X) = H^2(X, \mathbb{Z})$ for a Calabi-Yau manifold.

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§1. PROOF OF THE MAIN THEOREM AND THE COROLLARY

Let us choose a marking $\tau : R^2 f_* \mathbb{Z}_{\mathcal{X}} \simeq \Lambda \times \Delta$, where $\Lambda = (\Lambda, (*, *))$ is a lattice of signature $(3, N - 1)$ and consider the period map

$$p : \Delta \rightarrow \mathcal{D} := \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) = \mathbb{P}^{N+1}.$$

This map p is defined by $p(t) = \tau_{\mathbb{C}}([\omega_{\mathcal{X}_t}])$ and is known to be holomorphic. We notice that p is not constant by our assumption and the local Torelli Theorem.

Let us consider all the primitive sublattices Λ_n ($n \in \mathcal{N}$) of Λ . Put $\Delta_n := \{t \in \Delta \mid \tau(NS(\mathcal{X}_t)) = \Lambda_n\}$. Then one has a decomposition $\Delta = \sqcup_{n \in \mathcal{N}} \Delta_n$. Since \mathcal{N} is countable but Δ is uncountable, there exists an element of \mathcal{N} , say 1, such that Δ_1 is uncountable. Since $p(t) \in \Lambda_1^\perp \otimes \mathbb{C}$ for all $t \in \Delta_1$ and since p is holomorphic, one has then:

$$p(\Delta) \subset \mathcal{D}' := \{[\omega] \in \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) = \mathbb{P}^n.$$

Here we regard $\mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C})$ as a linear subspace of $\mathbb{P}(\Lambda \otimes \mathbb{C})$ defined by $(\Lambda_1, *) = 0$. Set $\mathcal{S} := \Delta - \Delta_1$. Then, by the Lefschetz (1, 1)-Theorem, we also have that:

- (1) $\Lambda_1 \subset \tau(NS(\mathcal{X}_t))$ for all $t \in \Delta$;
- (2) $t \in \mathcal{S}$ if and only if there is a vector $v \in \Lambda - \Lambda_1$ such that $(v, p(t)) = 0$.

Since both $\Lambda - \Lambda_1$ and $\{t \in \Delta \mid (v, p(t)) = 0\}$ for each $v \in \Lambda - \Lambda_1$ are countable, \mathcal{S} is countable as well.

We shall show the density of \mathcal{S} , i.e. the fact that $\mathcal{S} \cap U \neq \emptyset$ for any sufficiently small disk U .

Claim 1. $\text{rank } \Lambda_1^\perp \geq 3$.

Proof. If $\text{rank } \Lambda_1^\perp \leq 2$, then $\Lambda_1^\perp \otimes \mathbb{R}$ is spanned by the images of the real and imaginary parts of a holomorphic 2-form. This implies that Λ_1^\perp is a positive definite lattice of rank 2 and that \mathcal{D}' consists of two points. However, the period map p is then constant, a contradiction. \square

Let us choose a holomorphic coordinate z of U centered at P . We also choose integral basis of Λ_1^\perp and write $p|_U$ as $p(z) = [1 : f_1(z) : f_2(z) : \cdots : f_n(z)]$ with respect to this basis. Here we have $n \geq 2$ by Claim 1. We may also assume that $f_1(z)$ is not constant. In what follows, for each $\vec{a} = (a_0, a_1, a_2, \dots, a_n) \in \mathbb{R}^{n+1} - \{\vec{0}\}$, we put:

$$f_{\vec{a}}(z) := a_0 + a_1 f_1(z) + a_2 f_2(z) + \cdots + a_n f_n(z);$$

$l_{\vec{a}} := a_0 x_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$, where $[x_0 : x_1 : \cdots : x_n]$ is the homogeneous coordinates of \mathbb{P}^n ; and

$H_{\vec{a}} := (l_{\vec{a}} = 0) \subset \mathbb{P}^n$, the hyperplane defined by the linear form $l_{\vec{a}}$.

Let k be an element of $\{2, \dots, n\}$. Since $\dim_{\mathbb{R}} \mathbb{C} = 2$, one then finds an element $(r_{0,k}, r_{1,k}, r_{2,k}) \in \mathbb{R}^3 - \{\vec{0}\}$ such that $r_{0,k} \cdot 1 + r_{1,k} f_1(0) + r_{2,k} f_k(0) = 0$ (*). Put $\vec{r}_k := (r_{0,k}, r_{1,k}, 0, \dots, 0, r_{2,k}, 0, \dots, 0)$. Here $r_{2,k}$ is located at the same position as x_k in $[x_0 : x_1 : \cdots : x_n]$. In this notation, the equality (*) is rewritten both as $p(0) \in H_{\vec{r}_k}$ and as $f_{\vec{r}_k}(0) = 0$.

Claim 2. $p(U)$ is not contained in $\cap_{k=2}^n H_{\vec{r}_k}$.

Proof. Assuming to the contrary that $p(U) \subset \cap_{k=2}^n H_{\vec{r}_k}$, we shall derive a contradiction. Since $f_1(z)$ is not constant, we have $r_{2,k} \neq 0$ for each k . Therefore $\cap_{k=2}^n H_{\vec{r}_k}$ is a line $L \simeq \mathbb{P}^1$ defined over \mathbb{R} in \mathbb{P}^n . This leads the same contradiction as in Claim 1. \square

By Claim 2, there exists k such that $p(U) \not\subset H_{\vec{r}_k}$, i.e. $f_{\vec{r}_k}(z) \not\equiv 0$. Since $f_{\vec{r}_k}(0) = 0$, we may choose a small circle $\gamma \subset U$ around $z = 0$ such that $f_{\vec{r}_k}(z)$ has no zeros on γ . Set $K := \min\{|f_{\vec{r}_k}(z)| \mid z \in \gamma\}$ and $M := \max\{|f_i(z)| \mid z \in \gamma, i = 0, 1, \dots, n\}$, where we define $f_0(z) \equiv 1$. Note that $K > 0$ and $M > 0$. Then, by using the triangle inequality, we see that $|f_{\vec{r}_k}(z) - f_{\vec{r}_k}(0)| \leq |f_{\vec{r}_k}(z)|$ on γ , provided that

$|\vec{a} - \vec{r}_k| < KM^{-1}(n+1)^{-1}$. Denote by V the open disk such that $\partial V = \gamma$. Then, by the Rouché Theorem, the cardinalities of zeros (counted with multiplicities) on V are the same for $f_{\vec{r}_k}$ and $f_{\vec{a}}$. In particular, $f_{\vec{a}}$ admits a zero on V . Since $\mathbb{Q}^{n+1} - \{\vec{0}\}$ is dense in $\mathbb{R}^{n+1} - \{\vec{0}\}$, one can then find an element $\vec{q} \in \mathbb{Q}^{n+1} - \{\vec{0}\}$ such that $f_{\vec{q}}(z)$ has a zero on V . Let us denote this zero by $Q \in V(\subset U)$. Then, $f_{\vec{q}}(Q) = 0$ and $p(Q) \in H_{\vec{q}}$. Recall that Λ_1^\perp is primitive in Λ , Λ is non-degenerate, and that our homogeneous coordinates $[x_0 : x_1 : \dots : x_n]$ are chosen by means of integral basis of Λ and the rational linear equations $(\Lambda_1.*) = 0$. Therefore one can find an element $0 \neq v \in \Lambda$ such that $H_{\vec{q}} = \{x \in \mathbb{P}^n | (v.x) = 0\}$. Since this v satisfies $(v.p(Q)) = 0$, one has $v \in \tau(NS(\mathcal{X}_Q))$. On the other hand, since $\vec{q} \neq \vec{0}$, we have $v \notin \Lambda_1$. Hence this Q satisfies $Q \in \mathcal{S} \cap U$. Q.E.D.

Remark. In general, given a non-constant holomorphic map $g : \Delta \rightarrow \mathbb{P}^n$, the condition $h(\Delta) \cap H_{\vec{a}} \neq \emptyset$ is not open around \vec{a} if this point satisfies $h(\Delta) \subset H_{\vec{a}}$. For example, the holomorphic map $h(z) = [1 : -1 : z]$ satisfies $h(\Delta) \subset H_{(1,1,0)}$ and $h(\Delta) \cap H_{(1-\epsilon,1,0)} = \emptyset$ for any $\epsilon \neq 0$. \square

Proof of Corollary. We shall show by the descending induction on the Picard number $\rho := \rho(F)$. Note that $0 \leq \rho(F) \leq N = B_2(F) - 2$. Let $u : \mathcal{U} \rightarrow \mathcal{K}$ be the Kuranishi family of F . This is a germ of the universal deformation of F and is of dimension N by $H^1(T_F) \simeq H^1(\Omega_F^1)$. Therefore, \mathcal{K} is realized as an open neighbourhood of $0 \in H^1(F, T_F)$ and is then assumed to be a small polydisk in \mathbb{C}^N . Then $R^2u_*\mathbb{Z}_{\mathcal{U}}$ is a constant system on \mathcal{K} . Choosing a marking $\tau : R^2u_*\mathbb{Z}_{\mathcal{U}} \simeq \Lambda \times \mathcal{K}$, one can define the period map

$$p : \mathcal{K} \rightarrow \mathcal{D} := \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega.\omega) = 0, (\omega.\bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \simeq \mathbb{P}^{N+1}.$$

This p is a local isomorphism by the local Torelli Theorem and by the fact that $\dim \mathcal{K} = \dim \mathcal{D} = N$. Therefore we are allowed to identify \mathcal{K} with a small open set of the period domain \mathcal{D} (denoted again by \mathcal{D}).

Assume first that $\rho(F) = N$. By x_i ($1 \leq i \leq N$), we denote the elements in Λ corresponding to an integral basis of $NS(F)$. For each j such that $0 \leq j < N$, we define the sublocus $\mathcal{A} \subset \mathcal{K} = \mathcal{D}$ by the equations

$$(x_1.\omega) = (x_2.\omega) = \dots = (x_j.\omega) = 0,$$

and consider the induced family $\pi : \mathcal{S} \rightarrow \mathcal{A}$. Here \mathcal{A} is of dimension $N - j > 0$. Then, by the construction, we have $\rho(\mathcal{S}_a) = j$ for generic $a \in \mathcal{A}$ (or more precisely, for any element of the complement of the countable union of the hypersurfaces $(x.\omega) = 0$, where x runs through the elements of $\Lambda - \mathbb{Z}\langle x_i | 1 \leq i \leq j \rangle$), and we are done.

We next assume that $\rho := \rho(F) < N$. Let us choose an integral basis x_1, x_2, \dots, x_ρ of $\tau(NS(F))$. Let us define the sublocus $\mathcal{B} \subset \mathcal{K} = \mathcal{D}$ by the equations

$$(x_1.\omega) = (x_2.\omega) = \dots = (x_\rho.\omega) = 0,$$

and consider the induced family $\pi : \mathcal{T} \rightarrow \mathcal{B}$. Here \mathcal{B} is of dimension $N - \rho > 0$. By the construction, π is not isotrivial and the fibers \mathcal{T}_b satisfy $\rho(\mathcal{T}_b) \geq \rho$. Then by the main Theorem, there is $b \in \mathcal{B}$ such that $\rho(\mathcal{T}_b) > \rho$. Now, by the descending induction on ρ , we are done. \square

Exploiting the same idea as in the Example in the Introduction, one also immediately obtains the following pretty

Corollary 2. *Set $GL^+(2, \mathbb{Q}) := \{M \in GL(2, \mathbb{Q}) | \det M > 0\}$. Let $w = \varphi(z)$ be a holomorphic function defined over a neighbourhood of $\tau \in \mathbb{H}$. Assume that $\varphi(\tau) \in \mathbb{H}$. Then, there exists a sequence $\{\tau_k\}_{k=0}^\infty \subset \mathbb{H} - \{\tau\}$ such that $\lim_{k \rightarrow \infty} \tau_k = \tau$ and that τ_k and $\varphi(\tau_k)$ are congruent for each k under the standard, linear fractional action of $GL^+(2, \mathbb{Q})$ on \mathbb{H} .*

Proof. Let us denote by E_w the elliptic curve of period w . Choose a small neighbourhood $\varphi(\tau) \in \Delta_2 \subset \mathbb{H}$. Then one has a family of elliptic curves $g : \mathcal{G} \rightarrow \Delta_2$ with the level two structure such that $\mathcal{G}_w = E_w$. Since φ is holomorphic, we may also choose a small neighbourhood $\tau \in \Delta \subset \mathbb{H}$ such that $\varphi(\Delta) \subset \Delta_2$ and that there exists a family of elliptic curves $h : \mathcal{H} \rightarrow \Delta$ similar to g . By pulling back $g : \mathcal{G} \rightarrow \Delta_2$ by φ and taking the fiber product, one obtains a family of abelian surfaces $a : \mathcal{A} := \mathcal{H} \times_\Delta \varphi^* \mathcal{G} \rightarrow \Delta$. Then, taking a crepant resolution of the quotient of \mathcal{A} by the inversion, we obtain a family of Kummer surfaces $f : \mathcal{X} \rightarrow \Delta$. By construction, one has $\mathcal{X}_z = \text{Km}(E_z \times E_{\varphi(z)})$.

Claim. *This family f is not trivial.*

Proof. If f is a trivial family, then there exists a K3 surface S such that $S \simeq \text{Km}(E_z \times E_{\varphi(z)})$ for all $z \in \Delta$. Note that the Kummer surface structures on S , that is, the isomorphism classes of abelian surfaces A such that $S \simeq \text{Km}(A)$, are determined by the choices of 16 disjoint smooth rational curves on S . Recall also that there are at most countably many smooth rational curves on a K3 surface. Then, there exist at most countably many isomorphism classes of such A . Denote all of them by A_i ($i \in \mathbb{N}$) and set $A_i = \mathbb{C}^2 / \Lambda_i$. For each A_i , the product structures on A_i , i.e. the structures of decompositions $A_i = E_i \times F_i$, are also countably many, because subtori of A_i are determined by the choices of sublattices of Λ_i . Hence, there are at most countably many isomorphism classes of pairs (E, F) such that $S \simeq \text{Km}(E \times F)$. However, since the set of the isomorphism classes of E_z ($z \in \Delta$) are uncountable, our family $f : \mathcal{X} \rightarrow \Delta$ is then non-trivial. \square

Recall by [SM] that $\rho(\mathcal{X}_z) = 18$ if E_z and $E_{\varphi(z)}$ are not isogenous and that $\rho(\mathcal{X}_z) \geq 19$ if E_z and $E_{\varphi(z)}$ are isogenous. Then, by the main Theorem, there exists a dense subset $\mathcal{S} \subset \Delta$ such that $\rho(\mathcal{X}_s) \geq 19$ for $s \in \mathcal{S}$, i.e. that E_s and $E_{\varphi(s)}$ are isogenous if $s \in \mathcal{S}$. Therefore, any sequence in $\mathcal{S} - \{\tau\}$ converging to τ satisfies our requirement. \square

§2. APPLICATIONS FOR JACOBIAN K3 SURFACES

A Jacobian K3 surface is an elliptically fibered K3 surface $\varphi_0 : X \rightarrow \mathbb{P}^1$ with section O . Our interest in this section is the Mordell-Weil group $MW(\varphi_0) := H^0(\mathbb{P}^1, X^\#)$ of φ_0 , i.e. the group of sections of φ_0 and its behaviour under small deformation. We denote by $r(\varphi_0)$ the rank of $MW(\varphi_0)$. Note that $0 \leq r(\varphi_0) \leq 18$ for a Jacobian K3 surface [Sh1].

By a local one-dimensional family of Jacobian K3 surfaces, we mean a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{W} & \xrightarrow{\mathcal{O}} & \mathcal{X} \\ f \downarrow & & \downarrow \pi & & \downarrow f \\ & & \text{id} & & \text{id} \end{array}$$

such that $\varphi_t : \mathcal{X}_t \rightarrow \mathcal{W}_t$ is a Jacobian K3 surface with section \mathcal{O}_t for each $t \in \Delta$. We set $r(t) := r(\varphi_t)$.

A similar but slightly different jumping phenomenon is observed for $r(t)$:

Proposition (2.1). *Let $f = \pi \circ \varphi : \mathcal{X} \rightarrow \mathcal{W} \rightarrow \Delta$ be a non-trivial family of Jacobian K3 surfaces. Set $\mathcal{S}_r := \{t \in \Delta | r(t) = r\}$. Then, there exists a unique r_0 such that \mathcal{S}_{r_0} is dense and uncountable and that $\cup_{r > r_0} \mathcal{S}_r$ is dense and countable.*

Remark.

- (1) The proof below shows that the set $\cup_{r < r_0} \mathcal{S}_r$ is at most countable and has no accumulation point in Δ . Moreover, as will be observed in the next Example (2.2), there actually exists a case where the set $\cup_{r < r_0} \mathcal{S}_r$ is not empty. Therefore, the behaviour of the Mordell-Weil rank in a family is slightly different from that of the Picard number described in the main Theorem.
- (2) As another comparison, we remark that $r(t)$ for a family of rational Jacobian surfaces is lower semi-continuous; Therefore the behaviour is quite different from the case of Jacobian K3 surfaces described in (2.1). This lower semi-continuity is a direct consequence of the stability Theorem and the fact that the sections of rational Jacobian surface are (-1) -curves.
- (3) It would be interesting to study a similar question for a family of Jacobian surfaces of Kodaira dimension 1. \square

Proof. By taking a local trivialization, we may assume that $\mathcal{W} = \mathbb{P}^1 \times \Delta$. Let $\mathcal{D} \subset \mathbb{P}^1 \times \Delta$ be the discriminant locus of φ . For the explicit description of \mathcal{D} , let us consider the Weierstrass model of $\varphi : \mathcal{X} \rightarrow \mathcal{W}$ (defined by \mathcal{O}) and write the equation as $y^2 = x^3 + a(w, t)x + b(w, t)$, where w is the inhomogeneous coordinate of \mathbb{P}^1 and t is the coordinate of Δ . Then \mathcal{D} is defined by (the reduction of) the equation $4a(w, t)^3 + 27b(w, t)^2 = 0 - (*)$. By construction, both $a(w, t)$ and $b(w, t)$ are polynomials with respect to w . Therefore the restriction map $\pi|_{\mathcal{D}} : \mathcal{D} \rightarrow \Delta; (w, t) \mapsto t$ has at most finitely many such bad points $P \in \mathcal{D}$ that $\pi|_{\mathcal{D}}$ is not smooth at P . Denote by $\mathcal{T} \subset \Delta$ the set of the image of these bad points. This is then a finite set. In addition, since the type of non-multiple singular fibers are uniquely determined by the local monodromy, the singular fibers of the fibrations $\varphi_t : \mathcal{X}_t \rightarrow \mathbb{P}^1$ are independent of $t \in \Delta - \mathcal{T}$. Write them by T_i ($i = 1, \dots, n$) and denote by m_i the number of the irreducible components of T_i . Then, by Shioda's formula [Sh1], we have $r(\varphi_t) = \rho(\mathcal{X}_t) - 2 - \sum_{i=1}^n (m_i - 1)$ for $t \in \Delta - \mathcal{T}$. Now the result follows from the main Theorem. \square

Example (2.2). In this example, we shall construct a family $f = \pi \circ \varphi : \mathcal{X} \rightarrow \mathcal{W} \rightarrow \Delta$ of Jacobian K3 surfaces such that $\rho(0) = 20$ but $r(0) = 0$, and that $\rho(t) < 20$ but $r(t) > 0$ for generic t .

Let us start from a family of rational Jacobian surfaces (with rational double points) $h : \mathcal{Z} \rightarrow \mathbb{P}^1 \times \Delta_u \rightarrow \Delta_u$ defined by the Weierstrass equation $y^2 = x^3 + ux + s^5$. Here u is the coordinate of Δ and s is the inhomogeneous coordinate of \mathbb{P}^1 . Then either by the Néron algorithm or by a direct calculation, one can easily check the following fact: \mathcal{Z} is smooth; \mathcal{Z}_u ($u \neq 0$) is smooth and $\mathcal{Z}_u \rightarrow \mathbb{P}^1$ has singular fibers of Type I_1 over $4u^3 + 27s^{10} = 0$ and of Type II over $s = \infty$; and $\mathcal{Z}_0 \rightarrow \mathbb{P}^1$ has one singular point of type E_8 over $s = 0$ and has a singular fiber of Type II over

Then, by taking an appropriate finite covering $\Delta_v \rightarrow \Delta_u$ and a simultaneous resolution of the pull back family, we obtain a family of smooth rational Jacobian surfaces $g : \mathcal{Y} \rightarrow \mathbb{P}^1 \times \Delta_v \rightarrow \Delta_v$ such that $\mathcal{Y}_v \rightarrow \mathbb{P}^1$ ($v \neq 0$) has 10 singular fibers of Type I_1 and one singular fiber of Type II , and $\mathcal{Y}_0 \rightarrow \mathbb{P}^1$ has one singular fiber of Type II^* and one singular fiber of Type II . By Shioda's formula, one has then $r(v) = 8$ for $v \neq 0$ and $r(0) = 0$.

Let us choose large number M such that the divisor $s = M$ on $\mathbb{P}^1 \times \Delta_v$ does not meet the discriminant locus. This is possible by the description above. Let us take the double covering $\mathbb{P}^1 \times \Delta_t \rightarrow \mathbb{P}^1 \times \Delta_v$ ramified over $s = M$ and $s = \infty$ and consider the relatively minimal model $f : \mathcal{X} \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$ of the pull back family $f' : \mathcal{X}' \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$. Note that \mathcal{X}' is equi-singular along the preimage of the cuspidal points of fibers $(\mathcal{Y}_v)_\infty$ of $\mathcal{Y}_v \rightarrow \mathbb{P}^1$. Therefore, by the monodromy calculation, one finds that $f : \mathcal{X} \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$ is a smooth family of Jacobian K3 surfaces such that $\mathcal{X}_0 \rightarrow \mathbb{P}^1$ has two singular fibers of Type II^* and one singular fiber of Type IV ; and $\mathcal{X}_t \rightarrow \mathbb{P}^1$ ($t \neq 0$) has 20 singular fibers of Type I_1 and one singular fiber of Type IV . Note also that $r(t) \geq r(v) = 8$ for $t \neq 0$. On the other hand, again by Shioda's formula, one has $20 \geq \rho(t=0) = 2 + r(t=0) + (9-1) + (9-1) + (3-1)$. Therefore $\rho(0) = 20$ and $r(0) = 0$ for the central fiber of f . Moreover, this family $\mathcal{X} \rightarrow \Delta$ is not trivial as a family of K3 surfaces. (Indeed, otherwise, we have $\mathcal{X} \simeq \mathcal{X}_0 \times \Delta_t$. Since $\text{Pic}(\mathcal{X}_0)$ is discrete, \mathcal{X}_0 does not admit a family of elliptic fibrations varying continuously. Then, our family $\mathcal{X} \rightarrow \Delta$ must be also trivial as a family of elliptic fiber spaces. However, this contradicts the fact that the types of singular fibers of \mathcal{X}_0 and \mathcal{X}_t are different.) Hence, by the main Theorem, we have $\rho(t) < 20$ for generic t . \square

Remark. This example also shows that there is a case where the behaviour of $r(t)$ is not honestly accompanied with that of $\rho(t)$. \square

Next we apply our main Theorem to study the structure of the Mordell-Weil lattices of Jacobian K3 surfaces. Here the Mordell-Weil group $MW(\varphi)$ with Shioda's positive definite, symmetric bilinear form $\langle *, * \rangle$ is called the Mordell-Weil lattice [Sh2]. This lattice structure on $MW(\varphi)$ made the study of Mordell-Weil groups extremally rich [Sh3]. By the narrow Mordell-Weil lattice $MW^0(\varphi)$ we mean the sublattice of $MW(\varphi)$ of finite index consisting of the sections which pass through the identity component of each fiber [Sh2]. Contrary to the case of rational Jacobian surfaces, the isomorphism classes of both $MW(\varphi)$ and $MW^0(\varphi)$ for Jacobian K3 surfaces are no more finite ([OS], [Ni]) and the whole pictures of them does not seem so clear even now. Our interest here is to clarify certain relationships among all of the Mordell-Weil lattices of Jacobian K3 surfaces:

Theorem (2.3). *For any given Jacobian K3 surface $\varphi : J \rightarrow \mathbb{P}^1$ of rank $r := r(\varphi)$, there exists a sequence $\{\varphi_m : J_m \rightarrow \mathbb{P}^1\}_{m=r}^{18}$ of Jacobian K3 surfaces such that*

- (1) $\varphi_r : J_r \rightarrow \mathbb{P}^1$ is the original $\varphi : J \rightarrow \mathbb{P}^1$;
- (2) $r(\varphi_m) = m$ for each m ; and
- (3) *there exists a sequence of isometric embeddings:*

$$MW^0(\varphi)(= MW^0(\varphi_r)) \subset MW^0(\varphi_{r+1}) \subset \cdots \subset MW^0(\varphi_{17}) \subset MW^0(\varphi_{18}).$$

In particular, the narrow Mordell-Weil lattice of a Jacobian K3 surface is embedded into the Mordell-Weil lattice of some Jacobian K3 surface of rank 18. Conversely, for every sublattice M of the (narrow) Mordell-Weil lattice of a Jacobian K3 surface of rank 18, there exists a Jacobian K3 surface whose narrow Mordell-Weil lattice

contains M as a sublattice of finite index. Moreover, for each given M there are at most finitely many isomorphism classes of the Mordell-Weil lattices of Jacobian K3 surfaces which contains M as a sublattice of finite index.

Proof. First we shall show the existence of a sequence in the statement. We may assume that $r \leq 17$. Let us consider the Kuranishi family $k : (J \subset \mathcal{U}) \rightarrow (0 \in \mathcal{K})$ of J . This is a germ of the universal deformation of J and is known to be smooth of dimension 20. Therefore, \mathcal{K} is realized as an open neighbourhood of $0 \in H^1(J, T_J)$ and is then assumed to be a small polydisk in \mathbb{C}^{20} . Then $R^2k_*\mathbb{Z}_{\mathcal{U}}$ is a constant system on \mathcal{K} . Choosing a marking $\tau : R^2k_*\mathbb{Z}_{\mathcal{U}} \simeq \Lambda \times \mathcal{K}$, where $\Lambda = \Lambda_{K3}$, let us consider as before the period map

$$p : \mathcal{K} \rightarrow \mathcal{D} = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \simeq \mathbb{P}^{21}.$$

Since p is a local isomorphism for the same reason as before, we may identify \mathcal{K} with an open neighbourhood $\mathcal{U} \subset \mathcal{D}$ of $p(0)$ by p . Since our argument is completely local, by abuse of notation, we write this \mathcal{U} again by \mathcal{D} and identify therefore $\mathcal{K} = \mathcal{D}$ by p .

Let us write a general fiber of $\varphi : J \rightarrow \mathbb{P}^1$ by E and choose an integral basis S_i ($i = 1, \dots, r$) of $MW^0(\varphi)$, where $r := r(\varphi)$. Then S_i and the zero section O are all non-singular rational curves and E is an elliptic curve such that $(S_i, E) = (O, E) = 1$. By the definition of the Mordell-Weil lattice $(MW(\varphi), \langle *, * \rangle)$, we have a (minus sign of) isometric injective homomorphism $\iota : MW^0(\varphi) \hookrightarrow NS(J)$ given by $S \mapsto S - O + ((O^2) - (S, O))E$ [Sh2]. Then, $\langle S_i, S_i \rangle = 4 + 2(S_i, O)$. Note also that E, O, S_i are linearly independent in $NS(J)$ [Sh1]. Let us consider $(r + 2)$ elements in Λ given by $e := \tau([E])$, $o := \tau([O])$ and $s_i := \tau([S_i])$. Then, these are also linearly independent in Λ .

Consider the subset \mathcal{L} of \mathcal{D} defined by $(e, *) = (o, *) = (s_i, *) = 0$. This is a smooth analytic subset of \mathcal{D} of dimension $20 - (r + 2) > 0$ and contains $p(0)$. Through the identification made above, we may regard $0 \in \mathcal{L} \subset \mathcal{K}$. Then we may speak of the family $\tilde{j} : \tilde{\mathcal{J}} \rightarrow \mathcal{L}$ obtained as the restriction of $k : \mathcal{U} \rightarrow \mathcal{K}$ to \mathcal{L} . Then by [Ko, Theorem 14] or by [Hu1, Section 1 (1.14)], one finds that \mathcal{L} is the locus on which the invertible sheaves $\mathcal{O}_J(E), \mathcal{O}_J(O), \mathcal{O}_J(S_i)$ on J lift to invertible sheaves \mathcal{E}, \mathcal{O} and \mathcal{S}_i on the whole space $\tilde{\mathcal{J}}$. Since $20 - (r + 2) \geq 1$ by $r \leq 17$, one can take a sufficiently small disk $0 \in \Delta \subset \mathcal{L}$ and obtains the induced family $j : \mathcal{J} \rightarrow \Delta$. We denote the restrictions of \mathcal{E}, \mathcal{O} and \mathcal{S}_i on \mathcal{J} by the same letter. We also shrink Δ freely whenever it is convenient. Note that $\chi(\mathcal{O}_J(S_i)) = 1$, $h^0(\mathcal{O}_J(S_i)) = 1$ and $h^q(\mathcal{O}_J(S_i)) = 0$ for $q > 0$, because S_i is a smooth rational curve on a K3 surface. Then by applying the upper semi-continuity of coherent sheaves and by the Theorem of cohomology, we see that $j_*\mathcal{S}_i$ are invertible sheaves which satisfy the base change property. Then $(j_*\mathcal{S}_i) \otimes \mathbb{C}(0) \simeq H^0(\mathcal{O}_J(S_i))$. Therefore, by Nakayama's Lemma, all of C_i lift not only as invertible sheaves but also as effective divisors on \mathcal{J} . By abuse of notation, we denote these divisors again by \mathcal{S}_i . Since the smoothness is an open condition for a proper morphism, $\pi|_{\mathcal{S}_i} : \mathcal{S}_i \rightarrow \Delta$ is also smooth. Combining this with the fact that small deformation of \mathbb{P}^1 is again \mathbb{P}^1 , we see that $S_{i,t} := \mathcal{S}_i|_{\mathcal{J}_t}$ is again a smooth rational curve on \mathcal{J}_t for all $t \in \Delta$. The same holds for $\mathcal{O}_t := \mathcal{O}|_{\mathcal{J}_t}$. Note that $\chi(\mathcal{O}_J(E)) = 2$, $h^0(\mathcal{O}_J(E)) = 2$ and $h^q(\mathcal{O}_J(E)) = 0$ for $q > 0$, because E is an elliptic curve on a K3 surface. Then, $h^q(\mathcal{E}|_{\mathcal{J}_t}) = 0$ and $h^0(\mathcal{E}|_{\mathcal{J}_t}) = 2$. Therefore, \mathcal{E} is a locally free sheaf of rank 2 which satisfies

the base change property. In particular, $j^*j_*\mathcal{E}|J = H^0(\mathcal{O}_J(E))$. Since $\mathcal{O}_J(E)$ is globally generated, we see again by Nakayama's Lemma that the natural map $j^*j_*\mathcal{E} \rightarrow \mathcal{E}$ is also surjective. Therefore we may take a morphism $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ over Δ associated to this surjection. Then, by the base change property, we find that the restriction $\Phi_t : \mathcal{J}_t \rightarrow \mathcal{W}_t$ coincides with the morphism given by the surjection $H^0(\mathcal{E}|\mathcal{J}_t) \otimes \mathcal{O}_{\mathcal{J}_t} \rightarrow \mathcal{E}|\mathcal{J}_t$. This is an elliptic fibration by $h^0(\mathcal{E}|\mathcal{J}_t) = 2$ and by the adjunction formula on a K3 surface. Therefore, the factorization $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ makes $j : \mathcal{J} \rightarrow \Delta$ a family of elliptic surfaces over Δ . By the invariance of the intersection number, we have $(\mathcal{S}_{i,t}, \mathcal{E}_t) = (S_i, E) = 1$. Therefore, $\mathcal{S}_{i,t}$ is also a section of Φ_t . The same holds for \mathcal{O}_t . Therefore $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ makes $j : \mathcal{J} \rightarrow \Delta$ a family of Jacobian K3 surfaces with zero section \mathcal{O} . Moreover, by passing to the Weierstrass family over Δ given by \mathcal{O} and using the characterization of $MW^0(\varphi)$ that $S \in MW(\varphi)$ is in $MW^0(\varphi)$ if and only if S does not meet the singular points of the Weierstrass model, one finds that $\mathcal{S}_{i,t}$ are all in $MW^0(\Phi_t)$. In addition, the intersection matrix of $\mathcal{E}_t, \mathcal{O}_t, \mathcal{S}_{i,t}$ are the same as the one for E, O, S_i in Λ and is then hyperbolic. Therefore, $\mathcal{E}_t, \mathcal{O}_t, \mathcal{S}_{i,t}$ are also linearly independent in $H^2(\mathcal{J}_t, \mathbb{Z})$. Hence so are in $NS(\mathcal{J}_t)$. Thus by the injection $MW^0(\Phi_t) \hookrightarrow NS(\mathcal{J}_t)$ quoted above, we see that $\mathcal{S}_{i,t}$ are also linearly independent in $MW^0(\Phi_t)$. In particular, $r(\Phi_t) \geq r$ for all $t \in \Delta$. Since the base space Δ is chosen in the Kuranishi space, our family $j : \mathcal{J} \rightarrow \Delta$ is not trivial. Therefore, by Proposition (2.1), there exists $t_0 \in \Delta$ such that $r(t_0) > r$. By the invariance intersection and by the relation between $\langle *, * \rangle$ and $(*, *)$ quoted above, we see that the map $a : MW^0(\varphi) \rightarrow MW^0(\Phi_{t_0})$ given by $S_i \mapsto \mathcal{S}_{i,t_0}$ is then an isometric injection.

If $r(t_0) = r + 1$, then we may define $\varphi_{r+1} : J_{r+1} \rightarrow \mathbb{P}^1$ to be this Jacobian K3 surface $\Phi_{t_0} : \mathcal{J}_{t_0} \rightarrow \mathbb{P}^1$.

Let us treat the case where $r(t_0) \geq r + 2$. Since $18 \geq r(t_0)$, we have $16 \geq r$. For simplicity, we abbreviate $\Phi_{t_0} : \mathcal{J}_{t_0} \rightarrow \mathbb{P}^1$ and $r(t_0)$ by $\varphi' : J' \rightarrow \mathbb{P}^1$ and r' respectively. We denote the image of the basis S_i ($1 \leq i \leq r$) of $MW^0(\varphi)$ in $MW^0(\varphi')$ by the same letters S_i and take $T_j \in MW^0(\varphi')$ $j = r + 1, r + 2, \dots, r'$ such that S_i and T_j form a basis of $MW^0(\varphi') \otimes \mathbb{Q}$ over \mathbb{Q} . (Here note that our embedding might not be primitive so that we can not prolong S_i to integral basis of $MW^0(\varphi')$ in general.) Let us consider the Kuranishi space \mathcal{K}' of J' and take the subspace $\mathcal{L}' \subset \mathcal{K}'$ defined by the fiber class E' of φ' , the zero section O , all of S_i , and T_{r+1} . Denote by $j' : \mathcal{J}' \rightarrow \mathcal{L}'$ the family induced by the Kuranishi family as before. Then, $\dim \mathcal{L}' = 20 - (2 + r + 1) > 0$, because E', O, S_i and T_{r+1} are linearly independent in $H^2(J', \mathbb{Z})$ and $r \leq 16$. In addition, considering \mathcal{L}' as a subspace in the period domain under the identification made as before, and applying the same argument as in the proof of the main Theorem, one finds that the Néron-Severi group of \mathcal{J}'_t for t being generic in \mathcal{L}' is isomorphic to the primitive closure of $\mathbb{Z}\langle E_{18}, O, S_i, T_{r+1} \rangle$ in $H^2(J', \mathbb{Z})$. In particular, $\rho(\mathcal{J}'_t) = r + 3$ for generic t . Moreover, by the same argument as above, one makes this family a family of Jacobian K3 surfaces $\mathcal{J}' \xrightarrow{\Phi'} \mathcal{W}' \rightarrow \mathcal{L}'$ such that each fiber $\Phi'_t : \mathcal{J}'_t \rightarrow \mathcal{W}'_t$ satisfies that $MW^0(\varphi) \subset MW^0(\Phi'_t)$ and that $r(\Phi'_t) \geq r + 1$. On the other hand, one has $r(\Phi'_t) \leq r + 1$ for generic t by Shioda's formula and by $\rho(\mathcal{J}'_t) = r + 3$. Then we have $r(\Phi'_t) = r + 1$ and may define $\varphi_{r+1} : J_{r+1} \rightarrow \mathbb{P}^1$ to be $\Phi'_t : \mathcal{J}'_t \rightarrow \mathcal{W}'_t$ for generic t . The first statement now follows from induction on $(18 - r)$.

Next we shall show the middle statement. Let $\phi' : S' \rightarrow \mathbb{P}^1$ be a Jacobian K3 surface such that $\text{rank}(t'_0) = 18$ and M a sublattice of $MW(t'_0)$. Then, by taking

a generic point of the locus of the Kuranishi space defined by the basis of M , zero section of ϕ' and general fiber of ϕ' , one gets a Jacobian K3 surface $\phi : S \rightarrow \mathbb{P}^1$ such that $M \subset MW^0(\phi)$ and $r(\phi) = r$.

Finally, we check the last assertion. Assume that $M \subset MW^0(\phi)$ and is of finite index. Since the pairing $\langle *, * \rangle$ is integral valued on $MW^0(\phi)$ [Sh2], one has $M \subset MW^0(\phi) \subset M^*$. Since $M \subset M^*$ is of finite index, the possibilities of $MW^0(\phi)$ is then only finitely many. By [Sh2], one has also $MW^0(\phi) \subset MW(\phi)/(\text{torsion}) \subset MW^0(\phi)^*$. Therefore each $MW^0(\phi)$ recovers $MW(\phi)/(\text{torsion})$ up to finitely many ambiguities. Now it suffices to check the boundedness of the torsion subgroups of Jacobian K3 surfaces. If the j -invariant is not constant, the result follows from the classification due to Cox [Co]. Let us consider the case where the j -invariant is constant. Note that a Jacobian K3 surface always admits at least one singular fiber, because its topological Euler number is positive. Therefore, by the classification of the singular fibers whose j -values are not ∞ and by the general fact that the specialization map $MW(\phi)_{\text{torsion}} \rightarrow (\phi^{-1}(t))_{\text{reg}}$ is injective, one can easily see that the possible torsion groups are at most $0, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$ or $(\mathbb{Z}/2)^{\oplus 2}$. Now we are done. \square

This Theorem coarsely reduces the study of $MW(\varphi)$ to those of the maximal rank 18. For further study, it might be worthwhile noticing that a Jacobian K3 surface with maximal Mordell-Weil rank 18 is necessarily “singular” in the sense of Shioda [SI] and that Nishiyama [Ni] has already constructed an infinite series of examples of such Jacobian K3 surfaces.

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